2 Definitions

1. Definitions:
   - **Vertex Set** \((V(G))\)
   - **Edge Family** \((E(G))\)
   - Edge \(\{v, w\}\) or \(vw\) joins \(v\) to \(w\)
   - **Simple graph**: Non-empty finite set \(V(G)\) and finite set \(E(G)\) of distinct unordered pairs of distinct elements of \(V(G)\)
   - **Graph**: allows loops and multiple edges; \(E(G)\) (edge family) is a finite family of unordered pairs of (not necessarily distinct) elements of \(V(G)\)
   - In this book, all graphs are finite and undirected, with loops and multiple edges allowed unless specifically excluded.
   - Two graphs \(G_1\) and \(G_2\) are **isomorphic** if we can pair up each vertex in \(V(G_1)\) with a vertex in \(V(G_2)\) in such a way that any two vertices in \(V(G_1)\) are connected by the same number of edges as their corresponding vertices in \(V(G_2)\).
     - *Example*: p. 15, pr. 2.3.
   - Graphs (vertices) may be ‘labelled’ or ‘unlabelled’
   - **Connected,Disconnected,Component,Union**
     - p.11 lists all connected unlabelled graphs with up to five vertices

2. Example: p. 15, pr. 2.5.

3. Other Definitions:
   - **Adjacent vertices** and a vertex **incident** to an edge
   - **Isolated vertex** (degree 0) and **end-vertex** (degree 1)
   - **Degree sequence** - degrees written in increasing order

4. **Handshaking Lemma** (Euler 1735): If several people shake hands, then the total number of hands shaken must be even, as two hands are involved in each handshake.
   - **Corollary**: In any graph, the number of vertices of odd degree is even.

5. More Definitions:
   - **Subgraph**
   - **Deletion**: \(G - v\)
   - **Contraction**: \(G \setminus e\)

6. Matrix representations: \(G\) is a graph with vertices \(\{1, 2, \ldots, n\}\).
   - **Adjacency matrix**: \(n \times n\) matrix whose \(ij\)-th entry is the number of edges joining vertex \(i\) to vertex \(j\).
   - **Incidence matrix**: \(n \times m\) matrix whose \(ij\)-th entry is 1 if vertex \(i\) is incident to edge \(j\), and 0 otherwise.
3 Examples

1. Examples:

- **Null graph** \((N_n)\): \(n\) vertices, no edges
- **Complete graph** \((K_n)\): \(n\) vertices, \(n(n - 1)/2\) edges

- **Regular graph**: every vertex has the same degree; **regular of degree** \(r\) or \(r\)-regular

- **Cycle graph** \((C_n)\): Connected, regular graph with \(n\) vertices

- **Path graph** \((P_n)\): \(n\) vertices, obtained by removing one edge from \(C_n\)

- **Wheel** \((W_n)\): \(n\) vertices, joining a vertex \(v\) to each vertex of \(C_{n-1}\)

- **Cubic graphs**: regular of degree 3;
- Example: **Petersen graph**: Star inside pentagon

- **Platonic graphs** (regular): tetrahedron, octahedron, cube, icosahedron, dodecahedron
- **Bipartite graph**: \(V(G)\) can be split into two disjoint sets \(A\) and \(B\) so that any edge in \(E(G)\) goes from a vertex in \(A\) to a vertex in \(B\).

- **Complete bipartite graph** \(K_{m,n}\): \(m + n\) vertices, \(mn\) edges

- **k-cube** \((Q_k)\): vertices correspond to sequence \((a_1, a_2, \ldots, a_k)\), where each \(a_i\) equals 0 or 1, and whose edges join those sequences that differ in one place; \(2^k\) vertices and \(k2^{k-1}\) edges, \(k\)-regular

- **Complement** \((\bar{G})\) of a simple graph

2. Examples to work out:

(a) p. 20, pr. 3.3
(b) p. 20, pr. 3.4
4 Three Puzzles

1. The Eight Circles Problem (8! = 40,320 possibilities)

- Place the letters $A, B, C, D, E, F, G, H$ into the eight circles in such a way that no letter is adjacent to a letter that is next to it in the alphabet.
- Easiest letters are $A$ and $H$, since there only next to one letter.
- Hardest circles are those in the middle, since there adjacent to six others.

2. Six People at a Party

*Show that, in any gathering of six people, there are either three people who all know each other or three people none of whom knows either of the other two.*

- Draw a graph with six vertices.
- Vertices are connected by a solid edge if the two people know each other.
- Vertices are connected by a dotted edge if the two people don’t know each other.
- Any vertex $v$ has degree 5. At least three of these edges must be of the same type. Assume dotted. Call the edges at the other end $w, x$, and $y$.
- If $w$ and $x$, $w$ and $y$, or $x$ and $y$ don’t know each other, respectively, then that edge would form the third edge of a dotted triangle. If, however, $x, y$, and $z$ all know each other then they form a solid triangle.

3. The Four Cubes Problem

*Given four cubes whose faces are coloured red, blue, green, and yellow, can we pile them up so that all four colours appear on each side of the resulting $4 \times 1$ stack?*

- Represent each cube by a graph with 4 vertices $R, B, G, and Y$.
- Two vertices are adjacent if and only if the cube has the corresponding colours on opposite faces.
- Superimpose the graphs to form a new graph $G$. (Number edges for cube)
- We need to find subgraphs $H_1$ (front and back) and $H_2$ (left and right) such that:
  a) Each subgraph contains exactly one edge from each cube: they tell us which colours appear front & back, left & right.
  b) The subgraphs have no edges in common: ensures that the faces on front & back are different than those on left & right.
  c) Each subgraph is 2-regular: ensures that each colour appears exactly once each on front, back, left, and right.