Chapter 6: Colouring Graphs

17 Coloring Vertices

1. Sections 17-19: Qualitative (Can we color a graph with given colors?)
   Section 20: Quantitative (How many ways can the coloring be done?)

2. Definitions:
   - If $G$ is a graph without loops, then $G$ is $k$-colorable if we can assign one of $k$ colors to each vertex in such a way that no two adjacent vertices have the same color.
   - If $G$ is $k$-colorable but not $(k-1)$-colorable, then the chromatic number ($\chi(G)$) of $G$ is $k$ ($G$ is $k$-chromatic).

3. Examples:
   - No upper bound on chromatic number of a general graph: $\chi(K_n) = n$.
   - $\chi(G) = 1$ if and only if $G$ is a null graph.
   - $\chi(G) = 2$ if and only if $G$ is a non-null bipartite graph.
   - Examples where $\chi(G) = 3$: $C_n$ and $P_n$, where $n$ is odd, Petersen graph
   - Examples where $\chi(G) = 4$: $W_n$, where $n$ is even

4. Theorem If $G$ is a simple graph with largest vertex degree $\Delta$, then $G$ is $(\Delta + 1)$-colorable.
   pf.
   - Induction on $|V(G)| = n$.
   - Let $G$ be a simple graph with $n$ vertices.
   - Delete any vertex $v$: $G' := G - v$ has $n - 1$ vertices and largest vertex degree at most $\Delta$.
   - By IH, $G'$ is $(\Delta + 1)$ colorable.
   - The vertices adjacent to $v$ have at most $\Delta$ colors. Color $v$ with any other color.

5. Theorem: (Brooks’ Theorem, 1941)
   If $G$ is a simple connected graph which is not a complete graph, and if the largest vertex-degree of $G$ is $\Delta(\geq 3)$, then $G$ is $\Delta$-colorable. (Proof in Section 18)

6. Both previous theorems are useful if vertex degrees are approximately the same.
   They tell little if the graph has a few vertices of large degree. (i.e.$K_{1,n}$)

7. Theorem Every simple planar graph is 6-colorable.
   pf.
   - Induction on $|V(G)| = n$. (Trivial for $n \leq 6$)
   - Let $G$ be a simple, planar, with $n$ vertices, and assume that all simple planar graphs with at most $n - 1$ vertices are 6-colorable.
   - By Theorem 13.6 (p. 68), $G$ contains a vertex, $v$, of degree at most 5.
   - $G' := G - v$ is thus 6-colorable.
   - Color $G$ with coloring of $G'$ and coloring $v$ with a color different from the (at most 5) adjacent vertices.
8. **Theorem:** Every simple planar graph is 5-colorable.

pf.

- Induction on \(|V(G)| = n\). (Trivial for \(n < 6\))
- Let \(G\) be a simple, planar, with \(n\) vertices, and assume that all simple planar graphs with at most \(n - 1\) vertices are 5-colorable.
- By Theorem 13.6 (p.68), \(G\) contains a vertex, \(v\), of degree at most 5.
- \(G' := G - v\) is thus 5-colorable.
- If \(\deg(v) < 5\), then \(v\) can be colored by any color not adjacent to \(v\).
- Assume \(\deg(v) = 5\), adjacent to vertices \(v_1, \ldots, v_5\).
- If \(v_1, \ldots, v_5\) are mutually adjacent, then \(G\) contains \(K_5\) as a subgraph, which is impossible since \(G\) is planar.
- Hence at least 2 vertices (WLOG \(v_1\) and \(v_2\)) are not adjacent.
- Contract edges \(vv_1\) and \(vv_2\). Result is planar with fewer than \(n\) vertices \(\Rightarrow\) 5-colorable.
- Now color \(v_1\) and \(v_2\) with the color originally assigned to \(v\) (w/ edges contracted).
- A 5-coloring of \(G\) is obtained by color \(v\) differently than the (at most 4) colors assigned to \(v_1, \ldots, v_5\).

9. **Theorem:** (Appel and Haken, 1976)
   Every simple planar graph is 4-colorable.

10. Example: (p. 86, pr. 17.7)
    Let \(G\) be a simple graph with \(n\) vertices, which is regular of degree \(d\). By considering the number of vertices that can be assigned the same color, prove that \(\chi(G) \geq n/(n - d)\).

11. Example: (p. 86, pr. 17.8) Let \(G\) be a simple planar graph containing no triangles.
    (a) Using Euler’s formula, show that \(G\) contains a vertex of degree at most 3.
    (b) Use induction to show that \(G\) is 4-colorable.
    (In fact, it can be proved that \(G\) is 3-colorable.)
19 Colouring Maps

1. The 4-color problem

- Whether a map can be colored with 4 colors so that no 2 adjacent countries are shown in the same color.
- Raised by Francis Gurthrie in 1852.
- Presented to the general public (London Mathematical Society) by Cayley in 1878.
- Kempe published an incorrect proof in 1879, modified by Heawood in 1890 into a proof of the five color theorem.
- First generally accepted proof by Appel and Haken in 1977 (builds on Kempe’s ideas).
  - First shows that every plane triangulation must contain at least one of 1,482 ‘unavoidable configurations.’
  - Second, a computer is used to show that each configuration is ‘reducible’, meaning that any plane triangulation containing such a configuration can be 4-colored by piecing together 4-colorings of smaller plane triangulations.
  - Together, these produce an inductive proof that all plane triangulations, and hence all planar graphs can be 4-colored.

Proof criticized, responded with 741 page long algorithmic version of their proof.

- Shorter proof more readily verifiable given by N. Robertson, D. Sanders, P.D. Seymour, and R. Thomas in 1997.

2. Definitions:

- A map is a 3-connected plane graph; it contains no cutsets with 1 or 2 edges, no vertices of degree 1 or 2.
- A map is \( k \)-colorable(f) if its faces can be colored with \( k \) colors with no adjacent faces having the same color.
- A graph is \( k \)-colorable(v) if its \( k \)-colorable, as in Section 17.

3. Theorem: Let \( G \) be a plane graph without loops, and let \( G^* \) be geometric dual of \( G \). Then \( G \) is \( k \)-colorable(v) \( \iff \) \( G^* \) is \( k \)-colorable(f).

pf. \( \Rightarrow \)

- Assume \( G \) is simple and connected \( \Rightarrow \) \( G^* \) is a map.
- Assuming we have a \( k \)-coloring(v) of \( G \), color each face of \( G^* \) with the color of the corresponding vertex in \( G \).
- No two adjacent faces of \( G^* \) have the same color because the vertices they correspond to in \( G \) are adjacent and have different colors.
- Thus, \( G^* \) is \( k \)-colorable(f).

\( \Leftarrow \)

- Suppose we have a \( k \)-coloring(f) of \( G^* \).
- \( k \)-color the vertices of \( G \) so that each vertex has the color of the face in \( G^* \) containing it.
- Again, no two adjacent vertices of \( G \) have the same color, and \( G \) is \( k \)-colorable.
4. Example: (p. 92, pr. 19.3)
Give an example of a plane graph that is both 2-colorable(f) and 2-colorable(v).

5. **Theorem:** A map $G$ is 2-colorable($f$) $\iff$ $G$ is an Eulerian graph.
pf. $\Rightarrow$ For every $v \in V(G)$, even # of faces at $v$ since covered with 2 colors.
$G$ is Eulerian since every vertex hence has even degree.
Alternate Proof in whole:
- By Exercise 15.9, $G$ is Eulerian $\iff G^*$ is bipartite.
- A connected graph without loops hence is 2-colorable $\iff$ it is bipartite.

6. **Corollary:** The four-color theorem for maps is equivalent to the four-color theorem for planar graphs.
Proof in book.

7. **Theorem:** Let $G$ be a cubic map. Then $G$ is 3-colorable($f$) $\iff$ each face is bounded by an even number of edges.
pf. $\Rightarrow$
- Given any face $F$, the faces surrounding $F$ must alternate in color.
- There must be an even number of them, so each face is bounded by even # of edges.
$\Leftarrow$
- We prove the dual result.
- Assume $G$ is a simple connected plane graph where each face is a triangle and every vertex has even degree ($\Rightarrow G$ is Eulerian).
- We must prove that $G$ is 3-colorable($v$) with colors $r, y, g$.
- By Theorem, $G$ Eulerian $\Rightarrow G$ is 2-colorable($f$), with colors black and whit.
- Color any white face so that $r, y, g$ appear in clockwise order, counter-clockwise black faces.
- Vertex coloring is extended to the whole graph.

8. **Theorem:** In order to prove the four-color theorem, it is sufficient to prove that each cubic map is 4-colorable($f$).
pf.
- Corollary above implies enough to show that 4-colorability($f$) of every cubic map $\Rightarrow$ 4-colorability($f$) of any map.
- Let $G$ be a map, and assume that every cubic map is 4-colorable($f$).
- Remove vertices of degree 2 without affecting coloring.
- Only remains to eliminate vertices of degree $\geq 4$.
- If $v$ has degree $n \geq 4$, then cover $v$ with an $n$-gon patch.
- Repeating this process for all such vertices, we obtain a cubic map that’s 4-colorable($f$) by hypothesis.
- 4-coloring of faces of $G$ is obtained by shrinking each patch to a single vertex and reinstating each patch of degree 2.
20 Colouring Edges

1. Definitions:
   - $G$ is $k$-colorable(e) (or $k$-edge colorable) if its edges can be colored with $k$ colors so that no two adjacent edges have the same color.
   - The chromatic index, $\chi'(G)$ is the number $k$ such that $G$ is $k$-colorable(e) but not $(k-1)$-colorable(e).

2. Theorem (Vizing, 1964)
   If $G$ is a simple graph with largest vertex-degree $\Delta$, then
   $$\Delta \leq \chi'(G) \leq \Delta + 1.$$

3. Chromatic index for particular graphs:
   - $\chi'(C_{2n}) = 2$, and $\chi'(C_{2n+1}) = 3$.
   - $\chi'(W_n) = n - 1$, if $n \geq 4$.
   - Example: (p. 95, pr. 20.5) Chromatic index of Platonic graphs?

4. Theorem: $\chi'(K_n) = n$ if $n$ is odd ($n \neq 1$), and $\chi'(K_n) = n - 1$ if $n$ is even.
   pf.
   - Assume $n \geq 3$. (Otherwise trivial)
   - If $n$ is odd, place the vertices as a regular $n$-gon.
   - Color $n$-cycle with a different color for each edge.
   - Color remaining edges with color of boundary edge parallel to it.
   - $K_n$ is not $(n - 1)$-colorable(e), as the largest number of edges of the same color is $(n - 1)/2$.
   - It follows that $K_n$ has at most $(n - 1)/2 \cdot \chi'(K_n) = (n - 1)^2/2$ edges.
   - If $n$ is even, $K_n$ can be built by attaching a new vertex to all vertices in $K_{n-1}$.
   - Color $K_{n-1}$ as before.
   - One color is missing at each vertex, and the colors are all different.
   - Color “new” edges of $K_n$ with the missing colors.
5. **Theorem:** The four-color theorem is equivalent to the statement that $\chi'(G) = 3$ for each cubic map $G$.

pf. $\Rightarrow$
- Since $G$ is cubic, each vertex is surrounded by a tetrahedron.
- Assume $G$ is 4-colored by $\alpha = (1,0)$, $\beta = (0,1)$, $\gamma = (1,1)$, and $\delta = (0,0)$.
- Construct a 3-coloring by coloring each edge $e$ by the sum of the colors of the two adjacent faces, mod 2. (Example)
- $\Delta$ cannot occur in edge coloring as 2 faces adjacent to each edge must have different colors.
- Furthermore, no two adjacent edges can share the same color.

$\Leftarrow$
- Suppose we have a 3-coloring for $G$: $\alpha$, $\beta$, $\gamma$ $\Rightarrow$ edge of each color at each vertex.
- Subgraph determined by edges of any pair of colors (i.e. $\alpha$ & $\beta$) is 2-regular, hence Eulerian.
- Theorem 19.1 (2-colorable $\iff$ Eulerian) implies that we can color its faces with two colors, 0 and 1.
- Doing this for each pair of colors, each edge is assigned two colors $(x, y)$, where each is 0 or 1.
- Since coordinates of two adjacent faces must differ in at least one place, $(1,0)$, $(0,1)$, $(1,1)$, $(0,0)$ give the required 4-coloring.

6. **Theorem:** (Kőnig 1916)
If $G$ is a bipartite graph with largest vertex-degree $\Delta$, then $\chi'(G) = \Delta$.
Proof in Book.

7. **Corollary:** $\chi'(K_{r,s}) = \max(r, s)$.

8. Example: (p. 95, pr. 20.7)
Prove that if $G$ is a cubic Hamiltonian graph, then $\chi'(G) = 3$. 
21 Chromatic Polynomials

1. \( P_G(k) = \# \) of ways to color the vertices of \( G \) with \( k \)-colors
   \( (P_G(k) \) is the chromatic function of \( G \), soon to be chromatic polynomial.\)

2. Chromatic Functions:
   - Determine \( P_G(k) \) for paths, trees, \( K_n \), \( C_n \).
   - In each case, determine how many ways they can be colored with 5 or 6 colors.
   - Observe that if \( k < \chi(G) \), then \( P_G(k) = 0 \), and \( k \geq \chi(G) \Rightarrow P_G(k) > 0 \).

3. Theorem: Let \( G \) be a simple graph, and let \( G - e \) and \( G \setminus e \) be the graphs obtained from \( G \) by deleting and contracting \( e \). Then
   \[
   P_G(k) = P_{G-e}(k) - P_{G\setminus e}(k).
   \]
   Give an example.
   \( \text{pf. (Book includes some } G/e \text{.)} \)
   - Equivalent to showing that \( P_{G-e}(k) = P_G(k) + P_{G\setminus e}(k) \).
   - Let \( e = vw \).
   - \# of \( k \)-colorings of \( G - e \) in which \( v \) and \( w \) have different colors is unchanged if \( e \) is added \( \Rightarrow \) equals \( P_G(k) \).
   - \# of \( k \)-colorings of \( G - e \) in which \( v \) and \( w \) have the same color is unchanged if \( v \) and \( w \) are identified with each other \( \Rightarrow \) equals \( P_{G\setminus e}(k) \).
   - Thus, \( P_{G-e}(k) = P_G(k) + P_{G\setminus e}(k) \).

4. Corollary: The chromatic function of a simple graph is a polynomial.
   \( \text{pf.} \)
   - Induction on the number of edges.
   - Basis: 0 edges \( \Rightarrow P_G(k) = k^n \), where \( n \) is the number of vertices (components).
   - Assume true for graphs with \( m \) or fewer edges, and let \( G \) have \( m + 1 \) edges.
   - \( G - e \) and \( G \setminus e \) have \( m \) edges, and hence \( P_{G-e}(k) \) and \( P_{G\setminus e}(k) \) are polynomials.
   - Result follows since \( P_G(k) = P_{G-e}(k) - P_{G\setminus e}(k) \).

Note: Quickly follows that \( G \) has \( n \) vertices \( \Rightarrow P_G(k) \) is of degree \( n \), and the coefficient of \( k^n \) is 1.

5. Example: (p. 99, pr. 21.4)
   (a) Prove that the chromatic polynomial of \( K_{2,s} \) is
   \[
   k(k-1)^s + k(k-1)(k-2)^s.
   \]
   (b) Prove that the chromatic polynomial of \( C_n \) is
   \[
   (k-1)^n + (-1)^n(k-1).
   \]