Section 2.3 - The Bolzano-Weierstrass Theorem

1. A number \( x \) is called a limit point (cluster point, accumulation point) of a set of real numbers \( A \) if \( \forall \epsilon > 0, (x - \epsilon, x + \epsilon) \) contains infinitely many points of \( A \).
   
   Note: A limit point need not be an element of the set, e.g. 0 is a limit point of \( \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \).

2. Note: When we say that we divide \([a, b]\) into two intervals of equal length, these will be \([a, \frac{a+b}{2}]\) and \([\frac{a+b}{2}, b]\).

3. Theorem 2-12 (Bolzano-Weierstrass Theorem): Every bounded infinite set of real numbers has at least one limit point.

   pf.
   
   - Let \( A \subset \mathbb{R} \) be bounded \( \Rightarrow \exists M > 0 \ni [-M, M] \).
   - Divide \([-M, M]\) into \([-M, 0]\) and \([0, M]\). (At least one \( A_1 \) has infinitely many points.)
   - Observe: (length of \( A_1 \)) = \( M = 2M/2 \).
   - Divide \( A_1 \) into two closed intervals of equal length. At least one, \( A_2 \), contains infinitely many points and has length \( 2M/2^2 = M/2 \).
   - Continue inductively so that for each \( k \in \mathbb{Z}^+ \), \( A_k \) is a closed interval of length \( 2M/2^k = M/2^{k-1} \) containing infinitely many points of \( A \).
   - But \( A_n \supset A_{n+1} \), and if \( A_n = [a_n, b_n] \), then \( \lim_{n \to \infty} (b_n - a_n) = \lim_{n \to \infty} \frac{M}{2^{n-1}} = 0 \).
   - Theorem 2-7 \( \Rightarrow \bigcap_{n=1}^{\infty} A_n \) has exactly one point, \( p \).
   - Let \( \epsilon > 0 \) be given. Choose \( N \) so that \( 2M/2^N = M/2^{N-1} < \epsilon \).
   - \( p \in A_N \), and length of \( A_N \) is \( M/2^{N-1} \Rightarrow \)
     \[ (p - \epsilon, p + \epsilon) \supset \left[ p - \frac{M}{2^{N-1}}, p + \frac{M}{2^{N-1}} \right] \supset A_N. \]
   - Hence, \( (p - \epsilon, p + \epsilon) \) contains infinitely many points of \( A \), since each \( A_n \) does.

4. Theorem 2-13: Let \( \{a_n\} \) be a sequence. Then \( L \) is a finite subsequential limit of \( \{a_n\} \iff L \)
   satisfies either of the following conditions:
   
   (i) There are infinitely many terms of \( \{a_n\} \) that equal \( L \).
   
   (ii) \( L \) is a limit point of the set consisting of the terms of \( \{a_n\} \).

   Sketch of proof.
   
   - Theorem 2-11 says \( L \) is a subsequential limit of \( \{a_n\} \iff (L - \epsilon, L + \epsilon) \) contains infinitely many terms of \( \{a_n\} \) for any \( \epsilon > 0 \).
   - This occurs \( \iff (i) \) or \( (ii) \) occurs.
5. Examples:

(a) Let \( \{a_n\} = \{(-1)^n\} = \{-1, 1, -1, 1, -1, \ldots\} \). The set \(-1, 1\) contains all terms in the sequence. Since \(-1\) and \(1\) appear infinitely many times, they are both subsequential limit points for \(\{a_n\}\). They are not limit points of \(\{a_n\}\), however, as \(-1, 1\) has no limit points.

(b) Example: \(\{27, 1, 27, \frac{1}{2}, 27, \frac{1}{3}, 27, \frac{1}{4}, 27, \frac{1}{5}, \ldots\}\): subsequential limits and limit points.

6. Theorem 2-14: Every bounded sequence has a convergent subsequence.

pf.

- Assume \(\{a_n\}\) is bounded. (Note: Cases not exclusive, but not Case 1 ⇒ Case 2.)
- Case 1: \(a_n = L\) for infinitely many terms, positions \(n_1 < n_2 < \ldots\).
- Then, the subsequence \(\{a_{n_1}, a_{n_2}, \ldots\}\) converges to \(L\).
- Case 2: Terms of sequence take on infinitely many distinct values.
- By the BW-Theorem, \(\exists p \ni (p - \epsilon, p + \epsilon)\) contains infinitely many \(a_n\) for any \(\epsilon > 0\).
- By Theorem 2-11, there hence is a subsequence of \(\{a_n\}\) that converges to \(p\).

7. Corollary 2-14: A bounded sequence that does not converge has more than one subsequential limit point.

Don’t prove.

8. Theorem 2-15:

(a) A sequence that is unbounded above has a subsequence that diverges to \(\infty\).

(b) A sequence that is unbounded below has a subsequence that diverges to \(-\infty\).

pf. of (a)

- Let \(\{a_n\}\) be a sequence that is unbounded above.
- We’ll inductively construct \(\{a_{n_k}\}\).
- Since \(\{a_n\}\) is unbounded above, there is a term, say \(a_{n_1}\) with \(a_{n_1} > 1\).
- There must be infinitely many terms of \(\{a_n\}\) that exceed 2. Otherwise, the largest term would be an upper bound.
- We hence can choose \(a_{n_2} \geq a_{n_2} > 2\) and \(n_2 > n_1\).
- Assume that we have found \(a_{n_k} \geq a_{n_k} > k\) and \(n_k > n_{k-1}\).
- There must be infinitely many terms of \(\{a_n\}\) that exceed \(k+1\). Pick \(a_{n_{k+1}} \geq a_{n_{k+1}} > k+1\) and \(n_{k+1} > n_{k}\).
- By induction, there hence is a subsequence \(\{a_{n_k}\}\) with \(a_{n_k} > k\), \(\forall k\).
- This subsequence \(\{a_{n_k}\}\) diverges to \(\infty\).

9. Theorem 2-16: A sequence \(\{a_n\}\) converges ⇔ it’s bounded and has exactly 1 subsequential limit.

Don’t prove.
10. Recall: We use convention that \( \pm \infty \) may be subsequential limit points.

11. Definition: Let \( \{a_n\} \) be a sequence of real numbers.
   - \( \limsup a_n = \lim a_n \) is the l.u.b. of the set of subsequential limit points of \( \{a_n\} \).
   - \( \liminf a_n = \lim a_n \) is the g.l.b. of the set of subsequential limit points of \( \{a_n\} \).

12. The l.u.b. (supremum) of the set of limit points of a sequence is a limit point, as is the g.l.b. (infimum). We hence can refer to \( \limsup a_n \) and \( \liminf a_n \) as the largest and smallest of the limit points of \( \{a_n\} \). (Not necessarily largest/smallest sequence values.)

13. Theorem 2-17: Let \( \{a_n\} \) be a bounded sequence of real numbers. Then:
   
   (a) \( \lim a_n = L \iff \forall \epsilon > 0, \) there are infinitely many terms of \( \{a_n\} \) in \( (L - \epsilon, L + \epsilon) \) but only finitely many terms of \( \{a_n\} \) with \( a_n > L + \epsilon \).
   
   (b) \( \lim a_n = K \iff \forall \epsilon > 0, \) there are infinitely many terms of \( \{a_n\} \) in \( (K - \epsilon, K + \epsilon) \) but only finitely many terms of \( \{a_n\} \) with \( a_n < K - \epsilon \).

   pf. of (a) \( \Rightarrow \)
   
   - Suppose \( \lim a_n = L \Rightarrow \) Theorem 2-11 \( \Rightarrow \) there are infinitely many terms of \( \{a_n\} \) in \( (L - \epsilon, L + \epsilon) \).
   
   - NTS Only finite number of terms of \( \{a_n\} \) that exceed \( L + \epsilon \) \( \forall \epsilon > 0 \).
   
   - (Contradiction): Assume \( \exists \epsilon > 0 \) \( \exists \) an infinite number of \( \{a_n\} \) exceed \( L + \epsilon \).
   
   - Since \( \{a_n\} \) is bounded about (say by \( M \)), \( \exists \) infinitely many terms between \( L + \epsilon \) and \( M \).
   
   - From these terms, construct a subsequence \( \{a_{n_k}\} \).
   
   - Since \( \{a_{n_k}\} \) is a bounded sequence, it has a convergent subsequence (Theorem 2.14), and since each term is \( \geq L + \epsilon \), this limit is \( \geq L + \epsilon \).
   
   - This implies that \( \exists \) a limit point of \( \{a_n\} \) larger than \( L \). C!

\( \Leftarrow \)

- Suppose \( \{a_n\} \) is a bounded sequence \( \exists \ \forall \epsilon > 0 \), \( (L - \epsilon, L + \epsilon) \) contains infinitely many terms of \( \{a_n\} \) and finitely many exceeding \( L + \epsilon \).

- Since \( (L - \epsilon, L + \epsilon) \) contains infinitely many terms of \( \{a_n\} \) \( \forall \epsilon > 0 \), \( L \) is a limit point of \( \{a_n\} \).

- Suppose \( M > L \Rightarrow M \) is not a limit point of \( \{a_n\} \). (Would mean \( L = \lim a_n) \)

- Let \( \epsilon = M - L \). Then \( (M - \epsilon, M + \epsilon) \) contains only finitely many terms of \( \{a_n\} \).

- Hence, \( M \) is not a limit point of \( \{a_n\} \).

14. Corollary: A bounded sequence \( \{a_n\} \) converges \( \iff \lim a_n = \lim a_n \).
15. Theorem 2-18: Let \( \{a_n\} \) and \( \{b_n\} \) be bounded sequences. Then

(a) \( \lim (a_n + b_n) \leq \lim a_n + \lim b_n. \)
(b) \( \lim (a_n + b_n) \leq \lim a_n + \lim b_n. \)

Note: \( \{a_n\} = \{(−1)^n\}, \{b_n\} = \{(−1)^{n+1}\} \) show that equality doesn’t always hold.

pf. of (a)

- Let \( K = \lim a_n \) and \( L = \lim b_n \), and let \( \epsilon > 0 \) be given.
- We need to show that \( \lim (a_n + b_n) \leq K + L. \)
- By Theorem 2.17, it is sufficient to show that at most finitely many terms of \( \{a_n + b_n\} \) exceed \( K + L + \epsilon \).
- Theorem 2.17 implies that there are at most finitely many \( \{a_n\} \) terms \( (m_1, \ldots, m_r) \) bigger than \( K + \frac{\epsilon}{2} \) and at most finitely many \( \{b_n\} \) terms \( (n_1, \ldots, n_s) \) bigger than \( L + \frac{\epsilon}{2} \).
- For any \( n \notin \{m_1, \ldots, m_r, n_1, \ldots, n_s\} \), we have
  \[ a_n + b_n < \left( K + \frac{\epsilon}{2} \right) + \left( L + \frac{\epsilon}{2} \right) = K + L + \epsilon. \]

16. We say that \( f \) is **bounded** if the range of \( f \) is a bounded set.

17. Let \( f \) and \( g \) be bounded functions with common domain \( D \). Then

(a) \( \text{l.u.b} (f + g) \leq \text{l.u.b}(f) + \text{l.u.b}(g). \)
(b) \( \text{g.l.b} (f + g) \geq \text{g.l.b}(f) + \text{g.l.b}(g). \)

pf. of (a)

- By Theorem 2-8, we can choose a sequence \( \{y_n\} \ni y_n \in \mathbb{R}(f + g), \ n = 1, 2, \ldots, \) and
  \[ \lim_{n \to \infty} y_n = \text{l.u.b.}(f + g). \]
- For each \( y_n \in \mathbb{R}(f + g), \exists x_n \in D(f + g) \ni \)
  \[ y_n = (f + g)(x_n) = f(x_n) + g(x_n). \]
- But \( f(x_n) \leq \text{l.u.b.}(f) \) and \( g(x_n) \leq \text{l.u.b.}(g), \forall n. \)
- Hence \( \text{l.u.b.}(f + g) = \lim_{n \to \infty} y_n \leq \text{l.u.b.}(f) + \text{l.u.b.}(g). \)

18. Example: Let

\[
f(x) = \begin{cases} 
0, & 0 \leq x \leq 1 \\
1, & 1 < x \leq 2,
\end{cases} \quad g(x) = \begin{cases} 
1, & 0 \leq x \leq 1 \\
0, & 1 < x \leq 2.
\end{cases}
\]

Then \((f + g)(x) = 1\) for \( 0 \leq x \leq 2 \). But \( \text{l.u.b.}(f) = 1 \) and \( \text{l.u.b}(g) = 1 \), so \( \text{l.u.b}(f + g) = 1 < 2 = \text{l.u.b.}(f) + \text{l.u.b}(g). \)