Section 4.1 - Forced Harmonic Oscillators

1. • Homogeneous - Right-hand side is zero
• Nonhomogeneous - Right-hand side is nonzero
• Forced harmonic oscillator (Forced equation - $f(t)$ is new, external force) $m > 0$, $k > 0$, $b \geq 0$

$$m \frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} + f(t)$$

$$\frac{d^2y}{dt^2} + \frac{b}{m} \frac{dy}{dt} + \frac{k}{m} y = f(t)$$

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + qy = g(t).$$

Second-order, linear, nonhomogeneous, nonautonomous.

•$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

is the corresponding homogeneous unforced equation.

2. Guess and Check method for solving $\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$.

Example: $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 0$.

Guess: $y(t) = e^{st} \Rightarrow s^2 + 6s + 8 = 0$.

This is just the characteristic polynomial, and hence the general solution is $y(t) = k_1e^{-2t} + k_2e^{-4t}$.

3. Extended Linearity Principle:

Consider a nonhomogeneous equation

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t)$$

and its corresponding homogeneous equation

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0.$$

• If $y_p(t)$ is a particular solution of the nonhomogeneous equation and $y_h(t)$ is a solution of the homogeneous equation, then $y_h(t) + y_p(t)$ is also a solution to the nonhomogeneous equation. (i.e. if $k_1y_1(t) + k_2y_2(t)$ is the general solution to the homogeneous equation, then $k_1y_1(t) + k_2y_2(t) + y_p(t)$ is a solution to the nonhomogeneous equation.)

• If $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation, then $y_p(t) - y_q(t)$ is a solution of the corresponding homogeneous equation. (i.e. Any solution to the nonhomogeneous equation can be written as $k_1y_1(t) + k_2y_2(t) + y_p(t)$.)

Hence, if $k_1y_1(t) + k_2y_2(t)$ is the general solution of the homogeneous equation, then

$$k_1y_1(t) + k_2y_2(t) + y_p(t)$$

is the general solution of the nonhomogeneous equation. (Verify the 2 bullet points.)
4. Finding the general solution for forced harmonic oscillators:

(1) Find the general solution of the homogeneous second-order equation.

(2) Find one particular solution of the nonhomogeneous second-order equation.

(3) Add the results of the previous two steps to obtain the general solution of the forced equation.

Observe that if the damping coefficient $p$ is positive (as in any physical, mechanical device), then the equilibrium point at the origin is a sink, and consequently all solutions approach $y_p(t)$ for large $t$.

\[
-p \pm \sqrt{p^2 - 4q}
\]

\[
p^2 - 4q < 0 \Rightarrow \text{spiral sink}
\]

\[
p^2 - 4q = 0 \Rightarrow \text{repeated negative eigenvalue (sink)}
\]

\[
p^2 - 4q > 0 \Rightarrow 2 \text{ distinct negative eigenvalues (sink)}
\]

5. Steady-state solution

- $y_p(t)$ is often called the **forced response**. ($y_p(t)$ need not be constant)
- The **steady-state response** describes the behavior of the forced response for large $t$.
- A solution to the unforced (homogeneous) harmonic oscillator is called the **natural response**.

6. Example: Method of Undetermined Coefficients:

Find the general solution of the differential equation

\[
\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2e^{-3t}.
\]

$(y_h(t) = k_1e^{-2t} + k_2e^{-4t}; \quad y_p(t) = -2e^{-3t})$

7. Example:

\[
\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = e^{-2t}
\]

- Observe that our initial guess $y_p(t) = e^{-t}$ does not work.
- As with repeated eigenvalue case, our next guess is $y(t) = kte^{-2t}$.
- We now determine $k$. ($k = 1/2$)