Section 2.4 - Analytic Methods for Special Systems

1. For systems of differential equations, special forms for which analytic techniques exist are rare. Hence, these special systems are valuable, and we can use them to develop intuition to use when studying systems for which analytic techniques are unavailable.

2. Consider the system

\[
\begin{align*}
\frac{dx}{dt} &= 2x + 2y \\
\frac{dy}{dt} &= x + 3y.
\end{align*}
\]

For the given functions \( Y(t) = (x(t), y(t)) \), check to see if \( Y(t) \) is a solution to the system.

(a) \((x(t), y(t)) = (2e^t, -e^t)\)

\[
\begin{align*}
x \text{ LHS} : \quad \frac{dx}{dt} &= 2e^t \\
x \text{ RHS} : \quad 2x + 2y &= 2(2e^t) + 2(-e^t) = 2e^t \\
y \text{ LHS} : \quad \frac{dy}{dt} &= -e^t \\
y \text{ RHS} : \quad x + 3y &= 2e^t + 3(-e^t) = -e^t
\end{align*}
\]

Since both equations are satisfied, \((x(t), y(t)) = (2e^t, -e^t)\) is a solution to the system.

(b) \((x(t), y(t)) = (3e^{2t} + e^t, -e^t + e^{4t})\)

\[
\begin{align*}
x \text{ LHS} : \quad \frac{dx}{dt} &= 6e^{2t} + e^t \\
x \text{ RHS} : \quad 2x + 2y &= 2(3e^{2t} + e^t) + 2(-e^t + e^{4t}) = 6e^{2t} + 2e^{4t} \\
y \text{ LHS} : \quad \frac{dy}{dt} &= -e^t + 4e^{4t} \\
y \text{ RHS} : \quad x + 3y &= 3e^{2t} + e^t + 3(-e^t + e^{4t}) = 3e^{2t} - 2e^t + 3e^{4t}
\end{align*}
\]

Since neither differential equation is satisfied, \((x(t), y(t)) = (3e^{2t} + e^t, -e^t + e^{4t})\) is not a solution to the system.

(c) \((x(t), y(t)) = (e^{4t} - 2e^t, e^{4t} + e^t)\)

\[
\begin{align*}
x \text{ LHS} : \quad \frac{dx}{dt} &= 4e^{4t} - 2e^t \\
x \text{ RHS} : \quad 2x + 2y &= 2(e^{4t} + e^t) + 2(e^{4t} - 2e^t) = 4e^{4t} - 2e^t \\
y \text{ LHS} : \quad \frac{dy}{dt} &= 4e^{4t} + e^t \\
y \text{ RHS} : \quad x + 3y &= e^{4t} - 2e^t + 3(e^{4t} + e^t) = 4e^{4t} + e^t
\end{align*}
\]

Since both differential equations are satisfied, \((x(t), y(t)) = (e^{4t} - 2e^t, e^{4t} + 3e^t)\) is a solution to the system.
3. A system of differential equations is said to be **decoupled** if the rate of change of one or more of the dependent variables depends only on its own value. If the equation for $\frac{dx}{dt}$ involves only $x$ and the equation for $\frac{dy}{dt}$ involves only $y$, we say that the system is **completely decoupled**, and we can solve the two equations separately.

4. **Example #1**: A completely decoupled system.
   Consider the system
   \[
   \frac{dx}{dt} = 3x, \quad \frac{dy}{dt} = -2y.
   \]
   Determine the general solution and the particular solution for which $Y(0) = (1, 4)$.
   **Solution:**
   The differential equation $\frac{dx}{dt} = 3x$ is an exponential model with general solution $x(t) = k_1e^{3t}$, and $\frac{dy}{dt} = -2y$ is an exponential model with general solution $y(t) = k_2e^{-2t}$. Hence, the general solution to the system is $(x(t), y(t)) = (k_1e^{3t}, k_2e^{-2t})$.
   Since $Y(0) = (x(0), y(0)) = (k_1, k_2) = (1, 4)$, we have $(x(t), y(t)) = (e^{3t}, 4e^{-2t})$.

5. If $\frac{dx}{dt}$ depends on both $x$ and $y$, but $\frac{dy}{dt}$ depends only on $y$, we say that $y$ decouples from the system and the system is **partially decoupled**. (If $\frac{dy}{dt}$ depends on both $x$ and $y$, but $\frac{dx}{dt}$ depends only on $x$, then $x$ decouples from the partially decoupled system.)

6. **Example #2**: A partially decoupled system.
   Consider the partially decoupled system
   \[
   \frac{dx}{dt} = 3x + 2y, \quad \frac{dy}{dt} = -y.
   \]
   (a) Derive the general solution to this system.
   **Solution:**
   In this case, $y$ decouples from the partially decoupled system, as $\frac{dy}{dt} = -y$ is an exponential model with general solution $y(t) = ke^{-t}$. Since we know that any solution to the system must have $y = ke^{-t}$, we can substitute this into the $\frac{dx}{dt}$ differential equation, which will leave the first-order linear differential equation
   \[
   \frac{dx}{dt} = 3x + 2ke^{-t} \quad \text{or} \quad \frac{dx}{dt} - 3x = 2ke^{-t}.
   \]
   Using the Section 1.9 method, we get $\mu = e^{\int -3dt} = e^{-3t}$. It follows that
   \[
   e^{-3t}\frac{dx}{dt} - 3e^{-3t}x = 2ke^{-t} \cdot e^{-3t} = 2ke^{-4t}
   \]
   \[
   \Rightarrow \frac{d}{dt} (e^{-3t}x) = 2ke^{-4t}
   \]
   \[
   \Rightarrow \int \frac{d}{dt} (e^{-3t}x) \ dt = \int 2ke^{-4t} \ dt = -\frac{1}{2}ke^{-4t} + C
   \]
   \[
   \Rightarrow x = e^{3t} \left( -\frac{1}{2}ke^{-4t} + C \right) = -\frac{1}{2}ke^{-t} + Ce^{3t}.
   \]
   The general solution hence is $Y(t) = (x(t), y(t)) = \left(-\frac{1}{2}ke^{-t} + Ce^{3t}, ke^{-t}\right)$.
(b) Determine the solutions that satisfy the initial conditions $Y(0) = (1, 0), Y(0) = (1, 1)$.

**Solution:**

Plugging $t = 0$ into our general solution, we obtain, $Y(0) = (-\frac{1}{2}k + C; k)$.

Hence, if $Y(0) = (1, 0)$, then the $y$-component implies $k = 0$, and the $x$-component requires $C = 1$.

$$Y(t) = (e^{3t}, 0)$$

If $Y(0) = (1, 1)$, then the $y$-component implies $k = 1$, and the $x$-component requires $\frac{1}{2} + C = 1$ or $C = \frac{3}{2}$.

$$Y(t)(-\frac{1}{2}e^{-t} + \frac{3}{2}e^{3t}, e^{-t})$$

7. Problem: Work through Problem 7 on p. 195. (The answer is in the back of the book.)

8. Section 2.3: **The Damped Harmonic Oscillator:**

The undamped harmonic oscillator equation is

$$m \frac{d^2y}{dt^2} = -ky,$$

where $m$ is the mass and $k$ is the spring constant. A damping force slows the motion, dissipating energy from the system. The form of the damping force is

$$-b \left( \frac{dy}{dt} \right),$$

where $b > 0$ is called the coefficient of damping (lumps together all damping forces such as friction, air resistance, etc.). To obtain the new model, we equate the product of mass and the acceleration with the sum of the spring force and the damping force, and we get

$$m \frac{d^2y}{dt^2} = -ky - b \frac{dy}{dt} \quad \text{or} \quad m \frac{d^2y}{dt^2} + b \frac{dy}{dt} + ky = 0.$$ 

Equation is called **damped harmonic oscillator**.

Let $p = b/m$ and $q = k/m$ to rewrite the equation as

$$\frac{d^2y}{dt^2} + p \frac{dy}{dt} + qy = 0.$$ 

We can convert this into a system by letting $v = \frac{dy}{dt}$ (velocity), and we have

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -qy - pv.$$
9. Guessing Solutions:
Consider the equation
\[
\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 15y = 0.
\]
We need \(y(t)\) whose second derivative can be expressed in terms of \(y\), \(\frac{dy}{dt}\), and constants.

Alternatively, we need \(y\), \(\frac{dy}{dt}\), and \(\frac{d^2y}{dt^2}\) to be similar enough that they cancel out each other.
We guess \(y(t) = e^{st}\). Substituting this into the LHS of the differential equation, we get
\[
\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 15y = (s^2 + 8s + 15)e^{st}.
\]
In order for \(y(t) = e^{st}\) to be a solution, this must be 0, and since \(e^{st}\) cannot be zero, we must have
\[
s^2 + 8s + 15 = (s + 3)(s + 5) = 0
\]
It follows that \(s = -3\) or \(s = -5\), and hence \(y_1(t) = e^{-3t}\) and \(y_2(t) = e^{-5t}\) are both solutions.

10. Problem: Work through problem 5 on p. 187. (The answer is in the back of the book.)